General Random Imperfections in the Buckling of Axially Loaded Cylindrical Shells

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Mean values as well as confidence intervals for the critical load of axially loaded cylindrical shells containing general random initial imperfections are obtained. The results display trends that have been observed experimentally and provide insight into previously proposed design criteria. It has been determined that nonaxisymmetric initial imperfections play a vital role in the determination of the critical load statistics and that this role varies with the shell radius-to-thickness ratio. The work should provide a sound theoretical basis for the design of randomly imperfect cylindrical shells.

Introduction

A CENTRAL difficulty in most buckling analyses of imperfection sensitive structures is the choice of an appropriate imperfection form. In particular, there is the difficult option between deterministic and probabilistic concepts. Deterministic methods furnish a precise technique in the exploration of the mechanisms involved in imperfection sensitivity, whereas it appears that probabilistic methods provide the key to acceptable design criteria as well as the correct interpretation of experimental data.

Probabilistic methods of analysis were initiated by Bolotin,[1] who recognized that both applied loads and initial geometric imperfections were random quantities and that a correct solution could be obtained only in statistical terms. His work was followed by a number of general analyses[2-4] for single-mode systems. These papers investigated critical load/initial imperfection statistics as well as such concepts as reliability. In addition, several authors[5,6] have considered the influence of axisymmetric random imperfections in cylindrical shells, whereas Ref. 7 has presented an analysis for both axisymmetric and nonaxisymmetric random imperfections that satisfy an ergodicity hypothesis. Probabilistic design criteria for cylindrical shells[8,9] also have been proposed. Tennyson et al.[8] have proposed the measurement of a number of axial imperfection profiles of cylindrical shells and the use of the measured quantity in the asymptotic results of Ref. 5. On the other hand, based on an examination of a great deal of experimental work, Roorda[9] has suggested that the statistics of the initial imperfection depend on the radius-to-thickness ratio R/h of the shell. Both of these design concepts assume that the design imperfections are axisymmetric. This assumption is based on the hypothesis that axisymmetric imperfections are of primary importance. Furthermore, it has been suggested that it is impractical for a designer either to determine the statistics of a complete mapping of a shell surface or even to use such information if it were obtained.

The present paper considers the elastic stability of an axially loaded cylindrical shell containing general random initial imperfections. The problem is solved by reducing it to a set of sample problems by way of a simulation procedure and then determining the critical load for each sample problem. For the present purposes, results are obtained for the restricted case that all of the critical imperfection amplitudes are independent and identically distributed with zero mean. However, this assumption may be removed if desired.

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The intentions of the paper are twofold. The primary objective is the determination of critical load statistics and confidence levels when the shell contains a general random imperfection. Secondly, the relationship between the results for general and axisymmetric random imperfections will be examined. The second objective has a direct bearing on the validity of the proposed design criteria of Refs. 8 and 9. Both of the objectives should be of interest to designers.

Some Preliminary Results

The present work represents an extension of a previous paper[10] wherein it was demonstrated that the axisymmetric cylindrical shell analysis due to Koiter[11] could be expanded to include the most general possible deterministic imperfection within the realms of an asymptotic approach. The analysis outlined in Ref. 10 has an analogous probabilistic development if it is assumed that the initial imperfection is a smooth two-dimensional Gaussian random process. A smooth process is a process that, with probability one, possesses continuous sample derivatives. Thus, the random potential energy of such a system can be viewed as a collection of deterministic functionals upon which has been induced a probability measure by way of the initial imperfection. It thus is permissible to define the conditions of stable and unstable equilibrium in a manner that is related directly to the corresponding deterministic criteria. Therefore, the random problem can be solved as a set of sample problems in a simulation procedure. This section outlines some of the features of Ref. 10 which are essential for the present analysis. For further details, one should refer to Ref. 10.

Consider a cylindrical shell containing random initial imperfections. Corresponding to the imperfect shell is a perfect shell that represents an idealized model of the real structure. The critical load of the model problem is called the classical or ideal critical load. This load is the least eigenvalue of the idealized problem and is represented by λ_I. The cylindrical shell is a unique problem, as the least eigenvalue is not related to a single eigenfunction, or, stated otherwise, the least eigenvalue is not related to a unique pair of axial and circumferential wave numbers, p and n, respectively. Rather, if the shell is assumed to be infinitely long (implying that p is continuous), then there are an extensive number of combinations of p and n which lead to the classical critical load. The appropriate critical axial wave numbers are designated as p_{n1} and p_{n2} for each value of n, $n=1, m$, where m is an integer depending on R/h and Poisson's ratio. The axial wave number corresponding to axisymmetric deformations is given a special symbol p_0.

The axial, circumferential, and radial displacements u, v, w then are expressed in terms of the eigenfunctions corresponding to the classical critical load. Upon substitution into the appropriate energy functional, a first-order approximation to the equilibrium equations is obtained as a set of $2+8m$ algebraic equations, which, after some manipulation, may be reduced to two algebraic equations in the variables a_0, b_0, which are the amplitudes of the axisymmetric modes. These equations are

$$-8R\lambda A_0-4R(\lambda-\lambda_I)a_0+3(8R\lambda)^2 \times\frac{[4R^2(\lambda-\lambda_I)^2-9(b_0^2-a_0^2)]}{[4R^2(\lambda-\lambda_I)^2-9(b_0^2+a_0^2)]^2}S_3 +18R(\lambda-\lambda_I)\frac{(8R\lambda)^2a_0}{[4R^2(\lambda-\lambda_I)^2-9(b_0^2+a_0^2)]^2}S_I -\frac{54(8R\lambda)^2a_0b_0}{[4R^2(\lambda-\lambda_I)^2-9(b_0^2+a_0^2)]^2}S_2=0 \tag{1a}$$

$$-8R\lambda B_0-4R(\lambda-\lambda_I)b_0-3(8R\lambda)^2 \times\frac{[4R^2(\lambda-\lambda_I)^2+9(b_0^2-a_0^2)]}{[4R^2(\lambda-\lambda_I)^2-9(b_0^2+a_0^2)]^2}S_2 +\frac{18R(\lambda-\lambda_I)(8R\lambda)^2b_0}{[4R^2(\lambda-\lambda_I)^2-9(b_0^2+a_0^2)]^2}S_I +\frac{54(8R\lambda)^2a_0b_0}{[4R^2(\lambda-\lambda_I)^2-9(b_0^2+a_0^2)]^2}S_3=0 \tag{1b}$$

where S_I, S_2, and S_3 depend on the nonaxisymmetric imperfections and are defined by

$$S_I=\sum_{n=I}^{m}\frac{I}{p_0^2}\left\{p_{nI}^2(A_{nI}^2+B_{nI}^2+C_{nI}^2+D_{nI}^2)+p_{n2}^2(A_{n2}^2+B_{n2}^2+C_{n2}^2+D_{n2}^2)\right\} \tag{2a}$$

$$S_2=\sum_{n=I}^{m}\frac{I}{p_0^2}\left\{n^2(A_{nI}A_{n2}+B_{nI}B_{n2}-C_{nI}C_{n2}-D_{nI}D_{n2})\right\} \tag{2b}$$

$$S_3=\sum_{n=I}^{m}\frac{I}{p_0^2}\left\{n^2(A_{nI}C_{n2}+A_{n2}C_{nI}+B_{nI}D_{n2}+B_{n2}D_{nI})\right\} \tag{2c}$$

where $\lambda=N/Eh$, and N is the axial load per unit circumference, and E is Young's modulus. Furthermore, the parameters A_0, B_0, $A_{nI,2}$,... represent the influence of the initial random imperfection. These quantities are random variables defined by

$$A_0=\frac{R}{2\pi l}\int_0^{l/R}\int_0^{2\pi}w_0(x,\theta)\sin p_0x\,d\theta\,dx \tag{3a}$$

$$B_0=\frac{R}{2\pi l}\int_0^{l/R}\int_0^{2\pi}w_0(x,\theta)\cos p_0x\,d\theta\,dx \tag{3b}$$

$$A_{nI}=\frac{R}{2\pi l}\int_0^{l/R}\int_0^{2\pi}w_0(x,\theta)\sin p_{nI}x\cos n\theta\,d\theta\,dx \tag{3c}$$

$$\vdots$$

and are referred to as the imperfection parameters. In these expressions, l is the length of the shell and $w_0(x,\theta)$ is the initial imperfection, which is a two-dimensional random process.

The stability criteria corresponding to the complete set of equilibrium equations can be derived explicitly, and they are presented in Ref. 10. The loss of stability in an imperfect cylindrical shell may result from either limit-point or bifurcational behavior. Limit-point buckling is by far the predominant case; however, special combinations of initial imperfections always lead to bifurcational behavior. In the present context, any bifurcation state that occurs for an imperfect shell will be termed a secondary bifurcation state, while the term primary bifurcation will refer to the bifurcation state of the perfect structure (model). There are two secondary bifurcation states that occur in a cylindrical shell which are of special interest. Furthermore, these secondary bifurcations may be isolated explicitly. The first of these was determined by Koiter[11] and refers to a bifurcation from axisymmetric to combined axisymmetric and nonaxisymmetric deflection states. In the present notation, this case arises when $S_I=S_2=S_3=0$ or when all of the nonaxisymmetric imperfection parameters vanish. The nondimensional bifurcational load is given by

$$\lambda_B=I+\frac{3}{2}\left(\frac{S_0}{R^2\lambda_I^2}\right)^{1/2}-\left[3\left(\frac{S_0}{R^2\lambda_I^2}\right)^{1/2}+9/4\left(\frac{S_0}{R^2\lambda_I^2}\right)\right]^{1/2} \tag{4}$$

where $S_0=A_0^2+B_0^2$. The second possibility occurs when $S_0=S_2=S_3=0$, that is, when there are no axisymmetric imperfection components and some of the nonaxisymmetric components vanish. The bifurcational load in this case is

$$\lambda_B=I+\frac{3}{\sqrt{2}}\left(\frac{S_I}{R^2\lambda_I^2}\right)^{1/2}-\left[\frac{6}{\sqrt{2}}\left(\frac{S_I}{R^2\lambda_I^2}\right)^{1/2}+\frac{9}{2}\left(\frac{S_I}{R^2\lambda_I^2}\right)\right]^{1/2} \tag{5}$$

The interesting feature to note is that both of these results are identical in form. Furthermore, they yield identical results if $S_0=2S_I$.

The foregoing bifurcation cases have been mentioned explicitly as they provide a very convenient method of determining the relative influences of different imperfection components in a simulation procedure. This will become apparent in the presentation of the final results.

Random Initial Imperfections

It was assumed at the outset that the initial imperfection $w_0(x,\theta)$ is a two-dimensional smooth Gaussian random process. Furthermore, Eqs. (3) specify the relationship between the random imperfection parameters A_0, B_0, $A_{nI,2}$, ..., and $w_0(x,\theta)$; however, before the analysis can proceed, the implications of Eqs. (3) must be investigated. In particular, the statistical description of A_0, B_0, $A_{nI,2}$,... is sought in terms of the statistics of $w_0(x,\theta)$.

The imperfection parameters A_0, B_0, $A_{nI,2}$,... and $w_0(x,\theta)$ are related linearly. That is, A_0, B_0, $A_{nI,2}$,... are linear functionals of $w_0(x,\theta)$. Thus, since $w_0(x,\theta)$ is a Gaussian process, it follows that A_0, B_0, $A_{nI,2}$,... are jointly Gaussian random variables. These random variables therefore are specified completely by their mean values, standard deviations, and correlation coefficients. The mean values are obtained directly from Eqs. (3) by taking the expected value of both sides of the relationship. As a result,

$$E[A_0]=\left(\frac{R}{2\pi l}\right)\int_0^{l/R}\int_0^{2\pi}E[w_0(x,\theta)]\sin p_0x\,d\theta\,dx \tag{6a}$$

$$E[B_0]=\left(\frac{R}{2\pi l}\right)\int_0^{l/R}\int_0^{2\pi}E[w_0(x,\theta)]\cos p_0x\,d\theta\,dx \tag{6b}$$

$$E[A_{nI}]=\left(\frac{R}{2\pi l}\right)\int_0^{l/R}\int_0^{2\pi}E[w_0(x,\theta)]\sin p_{nI}x\cos n\theta\,d\theta\,dx \tag{6c}$$

and so on. Furthermore, if we define the Fourier coefficients of $E[w_0(x,\theta)]$ as

$$b(\pm p,\pm n)=\left(\frac{R}{l\pi}\right)\int_0^{l/R}\int_0^{2\pi}E[w_0(x,\theta)] \times\frac{\sin}{\cos}(px)\,d\theta\,dx,\ n=0 \tag{7a}$$

$$b(\pm p, \pm n) = \left(\frac{2R}{l\pi}\right) \int_0^{l/R} \int_0^{2\pi} E[w_0(x,\theta)]$$

$$\times \begin{matrix}\sin\\ \cos\end{matrix}(px) \begin{matrix}\sin\\ \cos\end{matrix}(n\theta)\, \mathrm{d}\theta\, \mathrm{d}x, \quad n \neq 0 \qquad (7b)$$

where the plus or minus sign indicates sine or cosine, respectively, then

$$E[A_0] = \tfrac{1}{2} b(p_0, -0) \qquad (8a)$$

$$E[B_0] = \tfrac{1}{2} b(-p_0, -0) \qquad (8b)$$

$$E[A_{nl}] = \tfrac{1}{4} b(p_{nl}, -n) \qquad (8c)$$

and so on. The second-order statistics require the definition of the generalized spectral density. For the present purposes, it is defined as

$$S(\pm p_1, \pm p_2, \pm n_1, \pm n_2)$$

$$= \left(\frac{2R}{\pi l}\right)^2 \int_0^{l/R} \int_0^{l/R} \int_0^{2\pi} \int_0^{2\pi} E[w_0(x_1,\theta_1)\, w_0(x_2,\theta_2)]$$

$$\times \begin{matrix}\sin\\ \cos\end{matrix}(p_1 x_1) \begin{matrix}\sin\\ \cos\end{matrix}(p_2 x_2) \begin{matrix}\sin\\ \cos\end{matrix}(n_1 \theta_1) \begin{matrix}\sin\\ \cos\end{matrix}(n_2 \theta_2)$$

$$\times \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}x_1 \mathrm{d}x_2 \qquad (9)$$

where appropriate consideration is given to the factor $(2R/\pi l)$ for the cases in which n_1 or n_2 is zero. As in the previous definition, the plus or minus sign indicates sine or cosine, respectively. Consequently,

$$E[A_0^2] = \tfrac{1}{4} S(p_0, p_0, -0, -0) \qquad (10a)$$

$$E[A_0 B_0] = \tfrac{1}{4} S(p_0, -p_0, -0, -0) \qquad (10b)$$

$$E[A_0 A_{nl}] = \tfrac{1}{8} S(p_0, p_{nl}, -0, n) \qquad (10c)$$

$$\vdots$$

$$E[A_{nl}^2] = \frac{1}{16} S(p_{nl}, p_{nl}, n, n) \qquad (10d)$$

$$E[A_{nl}, A_{n2}] = \frac{1}{16} S(p_{nl}, p_{n2}, n, n) \qquad (10e)$$

and so on, where it should be recognized that

$$S(p_1, p_2, n_1, n_2) = S(p_1, p_2, n_2, n_1)$$

$$= S(p_2, p_1, n_2, n_1) = S(p_2, p_1, n_2, n_1)$$

The general form of the standard deviation follows directly from the preceding equations and is given by

$$\sigma_{pn} = \begin{cases} \tfrac{1}{4}[S(p, p, n, n) - b^2(p,n)]^{1/2}, & n \neq 0 \\ \tfrac{1}{2}[S(p, p, 0, 0) - b^2(p, 0)]^{1/2}, & n = 0 \end{cases} \qquad (11)$$

the correlation coefficient is specified by

$$\rho = \frac{S(p_1,p_2,n_1,n_2) - b(p_1,n_1)\, b(p_2,n_2)}{\mathfrak{D}} \qquad (12)$$

where

$$\mathfrak{D} = [S(p_1,p_1,n_1,n_1) - b^2(p_1,n_1)]^{1/2}$$

$$\times [S(p_2,p_2,n_2,n_2) - b^2(p_2,n_2)]^{1/2}$$

The number of parameters involved in the probabilistic analysis is quite extensive and depends on the radius-to-thickness ratio R/h, as well as Poisson's ratio ν of the shell. Table 1 demonstrates this interdependence for a number of values of R/h in the particular case in which Poisson's ratio is assumed as 0.3. With the information in Table 1 as well as the equilibrium equations and stability criteria, it is straightforward to simulate the probabilistic buckling problem.

Simulation Procedure

Experimental data that deal with the statistical aspects of initial imperfections are, in general, limited to simple structures such as struts and plates.[12] An extension of this type of work[13] has considered the comparison between simulated critical loads and experimental critical loads when the statistical aspects of the initial imperfections have been determined experimentally. Unfortunately, the preceding do not yield any significant information on the appropriate statistical description of initial imperfections in cylindrical shells.

Furthermore, the imperfection parameters are not entirely the appropriate measure of a random initial imperfection, as these parameters inherently include the influence of the shell thickness as well as some inconsistency in numerical factors due to the integration. This situation is remedied by considering an expansion of the initial random imperfection in terms of an asymptotically equivalent modal imperfection, that is, assuming

$$w_0(x,\theta) = \mu_0^a h \sin p_0 x + \mu_0^b h \cos p_0 x$$

$$+ \sum_{n=1}^{m} (\mu_{nl}^a h \sin p_{nl} x \cos n\theta + \mu_{nl}^b \sin p_{nl} x \sin n\theta$$

$$+ \mu_{nl}^c h \cos p_{nl} x \cos n\theta + \mu_{nl}^d \cos p_{nl} x \sin n\theta$$

$$+ \mu_{n2}^a h \sin p_{n2} x \cos n\theta + \mu_{n2}^b h \sin p_{n2} x \sin n\theta$$

$$+ \mu_{n2}^c h \cos p_{n2} x \cos n\theta + \mu_{n2}^d \cos p_{n2} x \sin n\theta) \qquad (13)$$

In this expression, μ represent the amplitude of a given imperfection component as a fraction of the shell thickness. All μ have a consistent physical interpretation. The relationships between the modal amplitudes and the imperfection parameters are thus

$$A_0 = \mu_0^a (h/2), \quad B_0 = \mu_0^b (h/2) \qquad (14a)$$

Table 1 Relation between radius-to-thickness ratio R/h and the number of statistical parameters in the analysis

R/h	m	Mean values, $2+8m$	Standard deviations, $2+8m$	Correlation coefficients, $(1+4m)(1+8m)$	Total, $(1+4m)(5+8m)$
20	4	34	34	561	629
50	6	50	50	1,225	1,325
100	9	74	74	2,701	2,849
1000	28	226	226	25,425	25,877

$$A_{nl,2}=\mu^a_{nl,2}(h/4),\ B_{nl,2}=\mu^b_{nl,2}(h/4) \tag{14b}$$

$$C_{nl,2}=\mu^c_{nl,2}(h/4),\ D_{nl,2}=\mu^d_{nl,2}(h/4) \tag{14c}$$

Thus the key imperfection quantities S_0, S_1, S_2, and S_3 become

$$S_0=\frac{h^2}{4}\left[(\mu^a_0)^2+(\mu^b_0)^2\right] \tag{15a}$$

$$S_1=\frac{h^2}{16p_0^2}\sum_{n=1}^{m}\Big\{p_{n1}^2\left[(\mu^a_{n1})^2+(\mu^b_{n1})^2+(\mu^c_{n1})^2+(\mu^d_{n1})^2\right]$$

$$+p_{n2}^2\left[(\mu^a_{n2})^2+(\mu^b_{n2})^2+(\mu^c_{n2})^2+(\mu^d_{n2})^2\right]\Big\} \tag{15b}$$

$$S_2=\frac{h^2}{16p_0^2}\sum_{n=1}^{m}n^2\left[\mu^a_{n1}\mu^a_{n2}+\mu^b_{n1}\mu^b_{n2}\right.$$

$$\left.-\mu^c_{n1}\mu^c_{n2}-\mu^d_{n1}\mu^d_{n2}\right] \tag{15c}$$

$$S_3=\frac{h^2}{16p_0^2}\sum_{n=1}^{m}n^2\left[\mu^a_{n1}\mu^c_{n2}+\mu^a_{n2}\mu^c_{n1}\right.$$

$$\left.+\mu^b_{n1}\mu^d_{n2}+\mu^b_{n2}\mu^d_{n1}\right] \tag{15d}$$

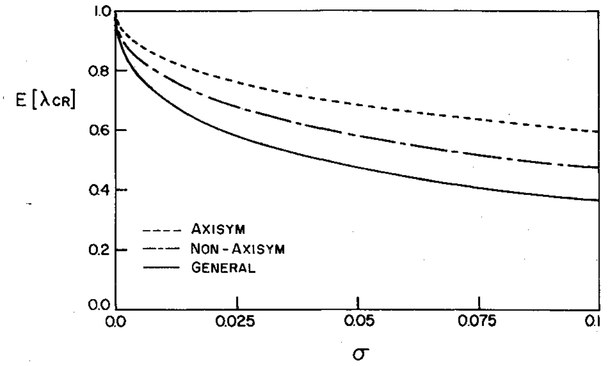
In passing, it should be noted that there is a significant difference in the weighting factors that determine the relative importance of the modal amplitudes in the expressions for S_0 and S_1. That is, in S_0 there is a factor $h^2/4$, whereas in S_1 there is a factor $h^2p_{n1}^2/16\,p_0^2$ or $h^2p_{n2}^2/16p_0^2$ ($p_{n1}<p_0$ and $p_{n2}<1/2\ p_0$ for all n). This demonstrates very clearly that, mode by mode, the axisymmetric imperfections should be predominant, and it thus verifies the usual assertion. However, such an argument completely ignores the fact that it is not the individual modal imperfection amplitudes that determine the imperfection sensitivity, but it is the actual value of S_0 or S_1. It is very conceivable that, in a comparison between S_0 and S_1, S_1 will be far more significant than S_0.

The statistical description of the modal imperfection amplitudes is very easy to determine. Because $w_0(x,\theta)$ was assumed to be Gaussian and there is a linear functional relationship between a given μ and $w_0(x,\theta)$, then all μ are jointly Gaussian random variables. The statistics of these quantities are specified by the statistics of $w_0(x,\theta)$. The present simulation procedure is therefore relatively straightforward. Appropriate values of μ^a_0, μ^b_0, $\mu^a_{nl,2}$,... are generated from a jointly Gaussian random number generator. These results then are used to generate sample values of A_0, B_0, S_1, S_2, and S_3. These values are, in turn, substituted into the deterministic equilibrium equations, which, in conjunction with the deterministic stability criterion, yield a sample critical load. This procedure is repeated until an appropriate number of sample critical loads has been generated. In the present case, the simulation proceeded on the assumption that all of the modal imperfection amplitudes were independent and identically distributed with zero mean. Thus the only significant statistic was the standard deviation of these parameters which was identical for all of the parameters. A thousand samples were taken for each simulation.

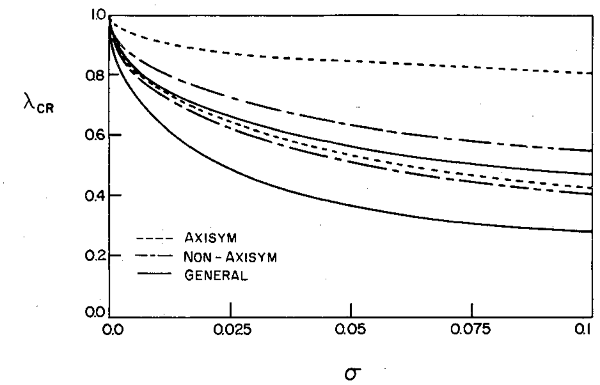
Discussion of Results

The results of a series of simulations are presented in Figs. 1-4. These are the specific cases for R/h ratios of 20, 100, 1000, and 4000, respectively, which represent situations for which the theoretical work is valid as well as situations that are of the physical interest.

Figures 1a, 2a, 3a, and 4a demonstrate the influence of the standard deviation of the modal imperfections σ on the expected value of the critical load. Three curves are present in

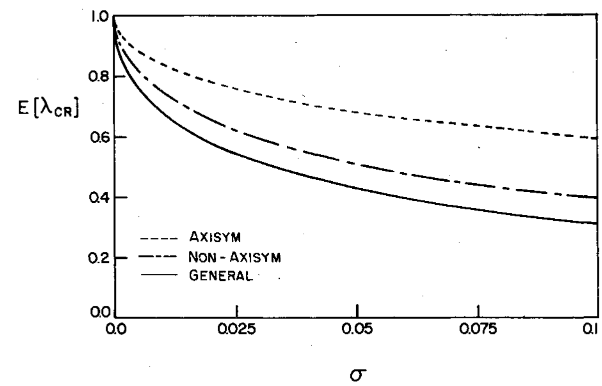


a) Mean critical loads

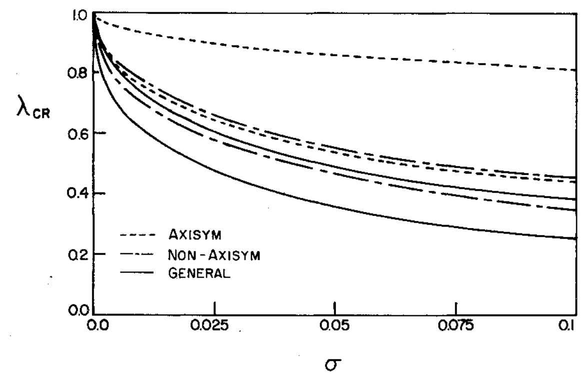


b) Upper and lower 99% confidence levels

Fig. 1 General (—), nonaxisymmetric bifurcation (– - – -), and axisymmetric bifurcation (----) results as a function of the input standard deviation σ ($R/h=20$).

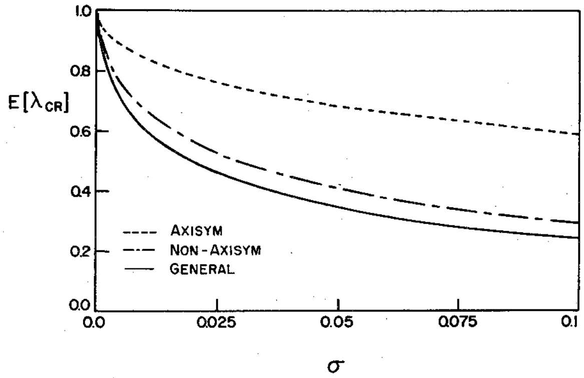


a) Mean critical loads

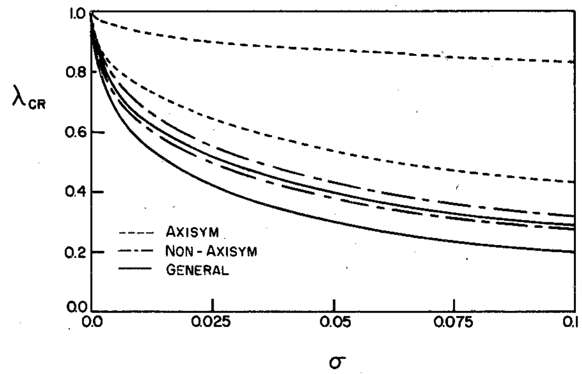


b) Upper and lower 99% confidence levels

Fig. 2 General (—), nonaxisymmetric bifurcation (—-—-), and axisymmetric bifurcation (----) results as a function of the input standard deviation σ ($R/h=100$).

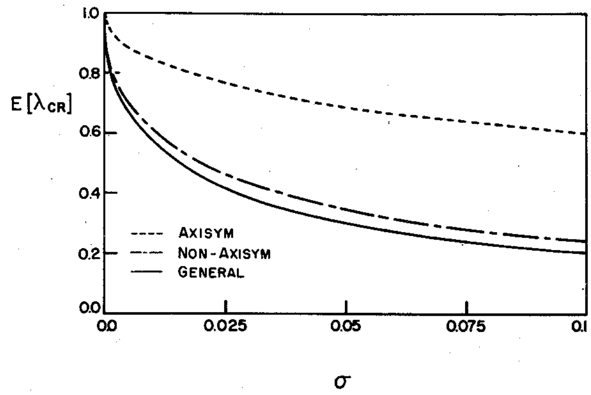


a) Mean critical loads

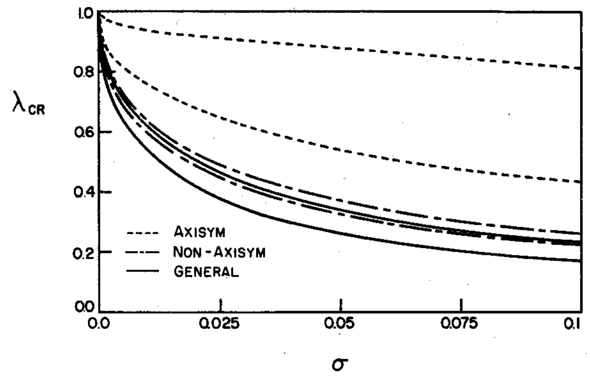


b) Upper and lower 99% confidence levels

Fig. 3 General (—), nonaxisymmetric bifurcation (—·—), and axisymmetric bifurcation (----) results as a function of the input standard deviation σ ($R/h = 1000$).



a) Mean critical loads



b) Upper and lower 99% confidence levels

Fig. 4 General (—), nonaxisymmetric bifurcation (—·—), and axisymmetric bifurcation (----) results as a function of the input standard deviation σ ($R/h = 4000$).

each figure: the solid line is the result for the general random imperfection, whereas the dot-dashed and dotted lines represent the cases of nonaxisymmetric and axisymmetric bifurcation, respectively. The latter two cases have been included for comparison and with the view to adopting these results as approximations in design situations. The significant feature in the results for the general random imperfection is the change in degree of imperfection sensitivity as R/h increases. The decrease in $E[\lambda_{cr}]$ is particularly pronounced for very small values of σ. Thus, as far as the expected value of the critical load is concerned, the imperfection sensitivity increases as R/h increases. A second very interesting (and perhaps surprising) result is that, as R/h increases, the axisymmetric and general imperfection results diverge. In contrast, the nonaxisymmetric and general imperfection results converge as R/h increases. In all cases, the nonaxisymmetric imperfection result is a better indicator of the general imperfection result than the axisymmetric. In fact, as R/h gets quite large, the nonaxisymmetric imperfection results are an excellent indicator of the general imperfection results.

The explanation of the preceding phenomenon follows directly from a consideration of the statistics of S_0, S_1, S_2, and S_3. From Eqs. (15) as well as the assumed statistics for μ_0^a, μ_0^b, $\mu_{nl,2}^a, \ldots$, it may be shown that the expected value of S_0 and the standard deviation of S_0 (axisymmetric imperfection) are, respectively,

$$E[S_0] = (h^2/2)\sigma^2; \quad \sigma_{S_0} = h^2\sigma^2/2$$

The equivalent results for S_1, S_2, and S_3 (nonaxisymmetric imperfections) are

$$E[S_1] = (h^2\sigma^2/4p_0^2)[mp_0^2 - \tfrac{1}{3}m(m+1)(2m+1)]$$

$$\sigma_{S_1} = (h^2\sigma^2/4\sqrt{2}p_0^2)[mp_0^4 - \tfrac{2}{3}p_0^2 m(m+1)(2m+1)$$

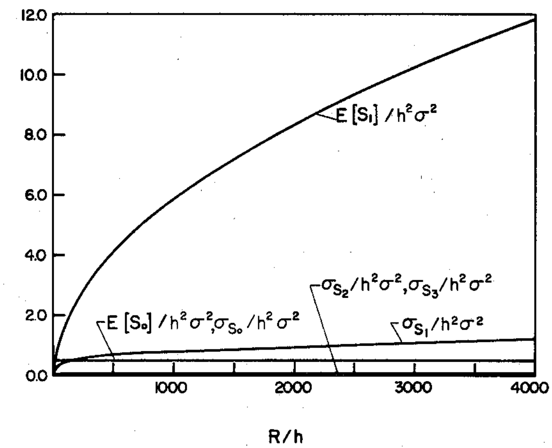


Fig. 5 Influence of the radius-to-thickness ratio R/h on the statistics of S_0, S_1, S_2 and S_3.

$$+ (m/15)(m+1)(2m+1)(3m^2+3m-1)]^{1/2}$$

$$E[S_2] = E[S_3] = 0$$

$$\sigma_{S_2} = \sigma_{S_3} = (h^2\sigma^2/8p_0^2)[(m/30)(m+1)$$

$$\times (2m+1)(3m^2+3m-1)]^{1/2}$$

The expected values and standard deviations of S_0, S_1, S_2, and S_3 are presented in Fig. 5 as a function of R/h. A glance at this figure immediately provides the answer to the phenomenon observed in the simulation. First, S_2 and S_3 play a minor role for all values of R/h. The expected values of these quantities are always zero, and the standard deviation is almost negligible. Thus it would appear that the buckling load for general imperfections depends primarily on the interaction between S_0 and S_1. The statistics of S_0 are independent of R/h, whereas the statistics of S_1 are monotone increasing with R/h and decrease to zero in the hypothetical situation that R/h approaches zero. Thus, for small values of R/h, S_0 plays the dominant role; for intermediate values, the interaction between S_0 and S_1 is important; and for large values of R/h, S_1 is predominant. For the range of validity of the present analysis, it appears that only the last two possibilities may be given theoretical justification.

Figures 1b, 2b, 3b, and 4b are plots of the upper and lower 99 percentiles of the critical loads for the three cases of general, axisymmetric, and nonaxisymmetric random imperfections. These curves appear to present two further anomalies: first, there is far more dispersion in the axisymmetric results than in the general or nonaxisymmetric results; and, secondly, the dispersion of the general and nonaxisymmetric results decreases as R/h increases.

Both of these points are explainable from a consideration of Fig. 5. The first follows because the expected value of S_1 is always much larger than its standard deviation. Thus, even for small values of σ, the expected value of S_1 is so large that the results are centered on a plateau of the critical load/initial imperfection relationship, and thus there is not a great deal of dispersion. This is not the case for axisymmetric imperfections as the expected value and standard deviation of S_0 are equal, and this yields a relatively large dispersion. The large expected value of S_1 is also the dominant factor for general imperfections, and it diminishes the dispersion of these results by a mechanism similar to that of the nonaxisymmetric case. The second feature results because the expected value of S_1 increases dramatically as R/h increases. Thus, the dispersion for the general and nonaxisymmetric case decreases. On the other hand, the statistics of S_0 are independent of R/h and therefore do not influence this phenomenon.

It should be noted, based on the previous observations, that a large dispersion of the results can occur only if there are a limited number of imperfection components present in the shells. Such a situation invalidates the present assumptions and leads immediately to an analysis of the type considered in Ref. 6.

Before continuing, it is worthwhile to back-track just slightly and discuss the hypotheses involved in the present analysis. First, within the realms of an asymptotic approach, the theoretical development is dependent only on the assumption that the initial imperfection is a smooth, two-dimensional random process. Since smoothness is to be expected in an actual shell, this assumption is not at all restrictive, and as a result the theoretical development previous to the simulation is quite general. On the other hand, in order that the simulation could be carried out relatively efficiently and in order that the results so obtained would be approachable for design purposes, some additional, more restrictive assumptions have been made. These are that the critical modal imperfection amplitudes μ_0^a, μ_0^b, $\mu_{nl,2}^a,\ldots$ are jointly Gaussian random variables that have zero mean, are statistically independent, and are distributed identically. It is not the intention here to defend these assumptions rigorously, as it is felt that there is insufficient experimental evidence to make such a strong statement. However, an attempt will be made to provide some justification, as well as to provide an assessment of the influence of these assumptions. The Gaussian assumption is one that has been invoked in a number of other analyses.[5,6] The justification for its use is the central limit theorem. The assumption of zero mean imperfection also has been adopted previously,[5,7] even though it has been shown that the inclusion of a mean imperfection[6] may alter the results significantly. The main justification for this assumption is that the perfect structure is the model of the real shell. Thus, it may be anticipated that the randomness in the shell is dispersed about the perfect form. The assumption that the modal imperfection amplitudes are statistically independent is perhaps a far more restrictive assumption than the previous two. It depends entirely on the autocorrelation function $E[\omega_0(x_1,\theta_1)\omega_0(x_2,\theta_2)]$ of the imperfection process in a set of cylindrical shells. There has been no experimental evaluation of the autocorrelation for a random imperfection in a cylindrical shell. However, in Ref. 14 it has been shown that a random process may be expanded in terms of a complete set of functions with random uncorrelated coefficients.

This is precisely the present situation, as the random imperfection is expanded in terms of the eigenfunctions of the linearized problem. It is therefore quite possible that the coefficients of this expansion μ_0^a, μ_0^b, $\mu_{nl,2}^a,\ldots$ are uncorrelated. It also may be noted that the influence of the correlation coefficient between imperfection amplitudes has been considered in some other multimode problems. For example, in the analysis of the two-mode buckling of a beam on an elastic foundation,[15] it was demonstrated that the correlation between the two critical imperfection parameters was not significant for systems that possessed a high degree of reliability. Also, in a more general treatment of two-mode problems,[16] the independence assumption was, in fact, used. An assessment of the influence of correlation between μ_0^a, μ_0^b, $\mu_{nl,2}^a,\ldots$ on the critical load statistics also may be obtained by considering the influence of this correlation on the statistics of S_0, S_1, S_2, and S_3. From the definition of S_0 and S_1, it follows that $E[S_0]$ and $E[S_1]$ are independent of the correlation coefficients of μ_0^a, μ_0^b, $\mu_{nl,2}^a,\ldots$ no matter what type of autocorrelation is assumed. The other statistical parameters, $E[S_2]$, $E[S_3]$, σ_{S_0}, σ_{S_1}, σ_{S_2}, σ_{S_3} are, however, influenced by the autocorrelation. The fact that $E[S_0]$ and $E[S_1]$ do not depend on the correlation is important; for, as soon as these quantities assume relatively large values, the statistics of the critical load are relatively insensitive to changes in the standard deviations of S_0 and S_1. This feature has been demonstrated for axisymmetric imperfections[6] and also is evident in the present results. That is, the dispersion of the critical load decreases as $E[\lambda_{cr}]$ decreases (or $E[S_1]$ increases) for a fixed value of σ. This feature is particularly noticeable if the axisymmetric and general results are compared. Such a comparison demonstrates that the large dispersion evident in the axisymmetric result is decreased greatly when combined with the nonaxisymmetric imperfections. This phenomenon may be attributed to an interaction between $E[S_0]$ and $E[S_1]$ which results in a "plateau"-like[17] region of lower imperfection sensitivity as $E[S_0]$ and $E[S_1]$ increase in magnitude. This characteristic plateau also is evident in the previous deterministic results.[10] These same results[10] also indicate that S_2 and S_3 do not play as significant a role as do S_0 and S_1. Thus, the changes in S_2 and S_3 due to the inclusion of the correlation coefficients will not be of importance. On the preceding basis, therefore, it is felt that the correlation between the imperfection amplitudes may be omitted.

The final assumption is that the amplitudes of the modal imperfections are distributed identically. As a result of the previous assumptions, this final hypothesis is equivalent to the view that all imperfection amplitudes have equal standard deviations. This assumption is an approximation that has been employed in order to facilitate the simulation and is based on the fact that it is the statistics of S_0, S_1, S_2, and S_3 which actually influence the statistics of the critical load. That is, since S_0, S_1, S_2, and S_3 involve sums of μ_0^a, μ_0^b, $\mu_{nl,2}^a,\ldots$, there is an inherent smoothing effect on the statistics of these latter parameters. The only experimental work that is related to this conjecture deals with the evaluation of modal amplitudes[18] and average modal amplitudes[19] as a function of wave number. Both of these references indicate that modal amplitudes decrease with increased wave number. In addition, Ref. 19 further implies that there are combinations of axial and circumferential wave numbers which yield equivalent imperfection amplitudes. Based on these papers, it perhaps may be inferred that the statistics of the imperfection amplitudes follow a similar pattern of wave number dependence. In that case, it follows that the standard deviations of μ_0^a, μ_0^b with wave number $(p_0, 0)$ and μ_{n1}^a, μ_{n1}^b, μ_{n1}^c, μ_{n1}^d with wave numbers (p_{n1}, n) will be of the same relative magnitude, whereas the standard deviation of μ_{n2}^a, μ_{n2}^b, μ_{n2}^c, μ_{n2}^d with wave numbers (p_{n2}, n) will be larger for small n and equivalent to the previous for larger values of n. It should be emphasized that the foregoing is only inference, and a great deal of weight is not placed upon it. It also must be emphasized that this assumption is an approximation that is taken for only those modal imperfections that are pertinent to the stability analysis. Otherwise, there is an inconsistency with the original assumption that the random imperfection $\omega_0(x, \theta)$ is a smooth process.

A final point of interest is the comparison of the present results with those of Refs. 8 and 9, as well as the implications of the present results for design purposes. It is fair to say that the present and previous results display common general trends. In particular, the present theoretical analysis displays the experimental trend observed in Ref. 9 that a large dispersion occurs for small values of R/h and that the dispersion decreases as R/h increases. In Ref. 9, this trend was accommodated by postulating an equivalent axisymmetric imperfection parameter whose statistics were a function of R/h. The present results yield a sound theoretical basis for this trend purely from the mechanisms involved in the phenomenon of instability. Moreover, this trend is dependent only on the nonaxisymmetric imperfection components. Qualitatively, however, the dispersion predicted in this analysis is not as large as that in the reported experiments, and it cannot be manipulated in the manner that was possible in Ref. 9. This discrepancy may result for a number of reasons. From the experimental point of view, an excessive dispersion may be attributed to the fact that the experimental results were an accumulation of a large number of different experiments with different materials under different controls. Thus, it seems reasonable to assume that a certain dispersion may be attributed to this variety in the experimental conditions. In contrast, a design and construction situation should occur under a more uniform set of conditions and thus lead to more credence of the theoretical predictions. The alternate implication is that only very few imperfection components occur in real shells which implies that the assumptions involved in the analysis are not valid. This, however, would appear to be a situation that could be rectified physically in a manufacturing process.

A comparison between the present results and those of Ref. 8 is not fruitful. This stems basically from the fact that the design curves in Ref. 8 yield "mean" critical loads, and, since axisymmetric imperfections are considered, there is no R/h dependence. On the other hand, the present results yield both mean critical loads and confidence intervals, both of which depend on R/h. However, it does seem that the philosophy of the design concept proposed in Ref. 8 is sound and that it can be extended easily to the present situation. The only difference between the previous and present methods is that all of the imperfection components must be considered for the evaluation of σ. Then, by specifying σ as well as R/h, it becomes straightforward to predict mean critical loads as well as confidence intervals from the present results.

The computational labor involved in obtaining the mean critical load and the confidence levels is considerable for general imperfections, and this leads to another useful result. That is, there is a great similarity of the general and nonaxisymmetric results, particularly as R/h increases. Furthermore, the nonaxisymmetric bifurcation result has been determined in closed form, and thus it is very easy to find a good first approximation to the general imperfection results.

Summary

The analysis of an axially loaded cylindrical shell with general random imperfections has been presented. Mean critical loads and confidence levels for the critical loads have been obtained in a simulation procedure, and these quantities are presented as functions of the radius-to-thickness ratio of the shell, as well as the statistics of the initial imperfection. It is suggested that the results may be useful for design purposes.

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